TOPIC FOR 2009 DECEMBER 09

SYMMETRY INVOLVING DIVISION WITH REMAINDER

How do you think about *division*? Do you think about it as the opposite of multiplication? Do you think about it as a fraction? How about a *decimal*? Maybe you think about it as something a calculator is good for! Indeed, we can input any two *rational* numbers into our calculator and out pops another rational number. Very handy, we can divide rational numbers (i.e. fractions) and get other rational numbers. What if we want to do the same thing with integers? Which integers can we divide by and *always* get an integer for an answer? Surely, we can divide $4 \div 2$ to get the integer 2, but $5 \div 2$ is not an integer. After convincing ourselves we see that the only integers satisfying the previous question are ± 1. Thus it seems like division requires us to think about fractions, but this is where *remainders* come in. Division with remainder is a simple process most easily explained with an example. Suppose we want to divide 127 by 5 using division with remainder. We first find the largest multiple of 5 which does not exceed 127, it is 125 which is $5 \cdot 25$. We then take the difference $127 - 125 = 2$ to find that the remainder is 2. Our answer then is that 127 divided by 5 is 25 with a remainder of 2. Below are two tables. The table on the left is just a multiplication table from one to six. The table on the right is the same multiplication table, except to get each entry for the right hand table we divide the corresponding entry in the left hand table by 7 and write the remainder. The entries in bold are the ones that were changed.

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

$\xrightarrow{\div \text{ by 7 and take remainder}}$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	**1**	**3**	**5**
3	3	6	**2**	**5**	**1**	**4**
4	4	**1**	**5**	**2**	**6**	**3**
5	5	**3**	**1**	**6**	**4**	**2**
6	6	**5**	**4**	**3**	**2**	**1**

The immediate response is to wonder why we divide by 7 and not some other number. To answer this try and think about all possible remainders when we divide all integers by 7. Convince yourself that no remainder can be larger than 6. As an exercise write down the ten by ten multiplication table and the corresponding table with each entry reduced by dividing by 11. Now try and look for symmetries in the tables you've written down. Which symmetries can you explain? Are any of the symmetries harder to explain?

We usually think about symmetry in geometry. For instance, a square has four lines of symmetry, a regular hexagon has six lines of symmetry, a circle has a line of symmetry for every α such that $0 \leq \alpha < 2\pi$. Here we begin to see that symmetry is not just a geometric idea, but also arises from division with remainder. In terms of non-geometric symmetry, this is just the tip of the iceberg. There is symmetry in the roots of polynomials, in the solution sets of algebraic equations, and in many other areas as well. The study of such symmetry is part of a large branch of mathematics known as *representation theory*. Representation theory is used everyday in physics and chemistry [1]. Deep questions about symmetry are of interest to many of today's mathematicians and scientists.

[1] Representation theory is so important in chemistry and crystallography that people in these disciplines must memorize *character tables* sometimes without having any idea about the mathematics behind their work.